JHEP HYPER VERSION

Received: *January 02, 2001*, Accepted: *February 20, 2001*

Self-dual Chern-Simons solitons in non-commutative space

Gustavo S. Lozano*

Departamento de Física, FCEyN, Universidad de Buenos Aires
Pab.1, Ciudad Universitaria, Buenos Aires, Argentina
E-mail: `lozano@df.uba.ar`

Enrique F. Moreno* and Fidel A. Schaposnik†

Departamento de Física, Universidad Nacional de La Plata C.C. 67
1900 La Plata, Argentina
E-mail: `emoreno@venus.fisica.unlp.edu.ar`,
`fidel@venus.fisica.unlp.edu.ar`

JHEP02(2001)036

Abstract: We construct exact soliton solutions to the Chern-Simons-Higgs system in non-commutative space, for non-relativistic and relativistic models. In both cases we find regular vortex-like solutions to the BPS equations which approach the ordinary selfdual non-topological and topological solitons when he non-commutative parameter θ goes to zero.

Keywords: Solitons Monopoles and Instantons, Chern-Simons Theories, Non-Commutative Geometry.

*Associated with CONICET
†Associated with CICBA

Contents

1. Introduction

After the connection between string theory and non-commutative field theories was unraveled [1]–[3], the study of solitons and instantons in non-commutative spacetimes has attracted much attention [4]–[18]. Chern-Simons (CS) theories in commutative space have played a central role for the understanding of relevant phenomena in planar physics [19, 20] and some of their properties started to be explored recently in the non-commutative case [21]–[27].

In ordinary $(2+1)$-dimensional space, models of relativistic and non-relativistic matter minimally coupled to gauge fields whose dynamics is governed by a CS term have self-dual vortex-like solutions [28]–[30]. It is then interesting to determine if this kind of solutions are also present in the non-commutative extension of these models. In this work we shall study the existence and properties of vortex-like solitons for Chern-Simons matter systems in non-commutative $(2+1)$-dimensional space. Our approach follows closely that developed in [18] for constructing exact non-commutative vortex solutions in the Maxwell-Higgs system except that now the dynamics of the gauge field is governed by a CS lagrangian. Also, we consider the non-relativistic case introduced in [28] in view of its relevance for studying Bohm-Aharonov effect and other interesting phenomena in planar physics.

After introducing the non-commutative models in section 2, we derive BPS equations and construct explicit solutions in the non-relativistic case in section 3. We find a family of non-topological BPS solitons parametrized by a constant f_0. The corresponding magnetic flux Φ is in general non quantized but becomes an integer in the $\theta \to 0$ limit. In section 4 we consider the relativistic case, and construct

BPS topological solitons with quantized magnetic flux which coincide with the regular ordinary vortex solutions when $\theta \to 0$. Finally, we present our conclusions in section 5.

2. The model

We consider 3-dimensional space-time with coordinates X^μ $(i = 0, 1, 2)$ obeying the following non-commutative relations

$$[X^\mu, X^\nu] = i\theta^{\mu\nu} \,. \tag{2.1}$$

The real antisymmetric matrix $\theta^{\mu\nu}$, can be brought into its canonical (Darboux) form by an appropriate orthogonal rotation

$$[X^1, X^2] = i\theta \,, \qquad [X^1, X^0] = [X^2, X^0] = 0 \,. \tag{2.2}$$

One way to describe field theories in non-commutative space is by introducing a Moyal product $*$ between ordinary functions. To this end, one can establish a one to one correspondence between operators $\hat{f}$ and ordinary functions f through a Weyl ordering

$$\hat{f}(X^1, X^2) = \frac{1}{2\pi}\int d^2k \tilde{f}(k_1, k_2) \exp\left(i(k_1X^1 + k_2X^2)\right) . \tag{2.3}$$

Then, the product of two Weyl ordered operators $\hat{f}\hat{g}$ corresponds to a function $f * g(x)$ defined as

$$f * g(x) = \exp\left(\frac{i\theta}{2}(\partial_{x_1}\partial_{y_2} - \partial_{x_2}\partial_{y_1})\right) f(x_1, x_2)g(y_1, y_2)\Big|_{x_1=x_2, y_1=y_2} . \tag{2.4}$$

Given a U(1) gauge field $A_\mu(x)$, the field strength $F_{\mu\nu}$ is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i(A_\mu * A_\nu - A_\nu * A_\mu) \,. \tag{2.5}$$

We shall couple the gauge field to a complex scalar field ϕ with covariant derivative

$$D_\mu\phi = \partial_\mu\phi - iA_\mu * \phi \,. \tag{2.6}$$

An alternative approach to non-commutative field theories which has shown to be very useful in finding soliton solutions [6] is to directly work with operators in the phase space (X^1, X^2), with commutator (2.2). In this case the $*$ product is just the product of operators and integration over the (X^1, X^2) plane is a trace,

$$\int dx^1 dx^2 f(x^1, x^2) = 2\pi\theta \,\mathrm{Tr}\, \hat{f}(X^1, X^2) \,. \tag{2.7}$$

In this framework, it is convenient to introduce complex variables z and $\bar{z}$

$$z = \frac{1}{\sqrt{2}}(x^1 + ix^2)\,, \qquad \bar{z} = \frac{1}{\sqrt{2}}(x^1 - ix^2) \tag{2.8}$$

and annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$ in the form

$$\hat{a} = \frac{1}{\sqrt{2\theta}}(X^1 + iX^2)\,, \qquad \hat{a}^\dagger = \frac{1}{\sqrt{2\theta}}(X^1 - iX^2) \tag{2.9}$$

so that (2.2) becomes

$$[\hat{a}, \hat{a}^\dagger] = 1\,. \tag{2.10}$$

In this way, through the action of $a^\dagger$ on the vacuum state $|0\rangle$, eigenstates of the number operator

$$\hat{N} = a^\dagger a \tag{2.11}$$

are generated. With this conventions, derivatives in the Fock space are given by

$$\partial_z = -\frac{1}{\sqrt{\theta}}[\hat{a}^\dagger,\ ]\,, \qquad \partial_{\bar{z}} = \frac{1}{\sqrt{\theta}}[\hat{a},\ ]\,. \tag{2.12}$$

After introducing

$$\hat{A}_z = \frac{1}{\sqrt{2}}(\hat{A}_1 - i\hat{A}_2)\,, \qquad \hat{A}_{\bar{z}} = \frac{1}{\sqrt{2}}(\hat{A}_1 + i\hat{A}_2) \tag{2.13}$$

the field strength and covariant derivatives take the form

$$\begin{aligned}
\hat{F}_{z\bar{z}} &= \partial_z \hat{A}_{\bar{z}} - \partial_{\bar{z}} \hat{A}_z - i[\hat{A}_z, \hat{A}_{\bar{z}}] \\
&= -\frac{1}{\sqrt{\theta}}\left([\hat{a}^\dagger, \hat{A}_z] + [\hat{a}, A_{\bar{z}}] + i\sqrt{\theta}[\hat{A}_z, \hat{A}_{\bar{z}}]\right) \equiv i\hat{B}\,, \\
D_{\bar{z}}\hat{\phi} &= \partial_{\bar{z}}\hat{\phi} - iA_{\bar{z}}\hat{\phi} = \frac{1}{\sqrt{\theta}}[\hat{a}, \hat{\phi}] - i\hat{A}_{\bar{z}}\hat{\phi}\,, \\
D_z\hat{\bar{\phi}} &= \partial_z\hat{\bar{\phi}} + i\hat{A}_z\hat{\bar{\phi}} = -\frac{1}{\sqrt{\theta}}[\hat{a}^\dagger, \hat{\bar{\phi}}] + i\hat{A}_z\hat{\bar{\phi}}
\end{aligned} \tag{2.14}$$

with $\hat{B}$ the magnetic field.

We will be interested in the non-commutative extension of the non-relativistic and relativistic Chern-Simons-matter systems introduced, in ordinary space, in refs. [28]–[30]. The gauge field dynamics for these models is governed by the Chern-Simons lagrangian $L_{\rm CS}[A]$ defined as

$$L_{\rm CS}[A] = \kappa\,\varepsilon_{\mu\nu\alpha}\left(A_\mu * \partial_\nu A_\alpha - \frac{2i}{3}A_\mu * A_\nu * A_\alpha\right). \tag{2.15}$$

The lagrangian for the non-commutative extension of the non-relativistic case will be taken as

$$L = L_{\rm CS}[A] + i\bar{\phi} * D_0\phi + \frac{1}{2}\overline{D_i\phi} * D_i\phi - \frac{1}{4}\lambda\phi * \bar{\phi} * \phi * \bar{\phi} \tag{2.16}$$

while for the relativistic case,

$$L = L_{\rm CS}[A] + \overline{D_\mu\phi} * D^\mu\phi - V[\phi * \bar\phi] \tag{2.17}$$

with V the sixth order potential

$$V[\phi * \bar\phi] = \frac{1}{\kappa^2}\phi * \bar\phi * \left(\phi * \bar\phi - v^2\right)^2 \tag{2.18}$$

taken at the selfdual point, where Bogomol'nyi equations can be found.

3. BPS equations for the non-relativistic case

The hamiltonian associated with lagrangian (2.16) is simply given by

$$H = \int d^2x \left(\frac{1}{2}\overline{D_i\phi} * D_i\phi + \frac{1}{4}\lambda\phi * \bar\phi * \phi * \bar\phi\right). \tag{3.1}$$

It can be written in the form

$$\begin{aligned} H = \int d^2x \Big(& -\frac{1}{2}\bar\phi(D_1 + i\alpha D_2)(D_1 - i\alpha D_2)\phi + \\ & + \phi * \bar\phi * \left(-\frac{\alpha}{2}F_{12} + \frac{1}{4}\lambda\phi * \bar\phi\right)\Big), \end{aligned} \tag{3.2}$$

where $\alpha = \pm 1$. We shall call $\alpha = -1$ the *selfdual* case and $\alpha = +1$ the *anti selfdual* one.

Using the Gauss law deriving from lagrangian (2.16),

$$\kappa\varepsilon_{ij}F_{ij} + \phi * \bar\phi = 0 \tag{3.3}$$

the hamiltonian takes the form

$$H = \int d^2x \left(-\frac{1}{2}\bar\phi(D_1 + i\alpha D_2)(D_1 - i\alpha D_2)\phi - \frac{1}{2}(\alpha + \lambda\kappa)\phi * \bar\phi * F_{12}\right). \tag{3.4}$$

Then, if the following relation among the two free parameters in the theory holds

$$\lambda\kappa = -\alpha \tag{3.5}$$

the lower bound for the hamiltonian is attained when the following Bogomol'nyi equations are satisfied

$$\begin{aligned} (D_1 - i\alpha D_2)\,\phi &= 0\,, \\ B &= -\frac{1}{2\kappa}\phi * \bar\phi\,. \end{aligned} \tag{3.6}$$

Let us first consider the selfdual ($\alpha = -1$) case. In operator language, eqs (3.6) can then be written as

$$
\begin{aligned}
D_{\bar{z}}\hat{\phi} &= 0\,, \\
B &= -\frac{1}{2\kappa}\hat{\phi}\hat{\bar{\phi}}\,.
\end{aligned}
\tag{3.7}
$$

In order to search for vortex solutions to these equations, we propose the ansatz

$$
\hat{\phi} = \sqrt{\frac{2|\kappa|}{\theta}}\sum_{n=0}^{\infty} f_n|n\rangle\langle n+M-1|\,, \tag{3.8}
$$

$$
\hat{A}_z = \frac{i}{\sqrt{\theta}}\sum_{n=0}^{\infty} d_n|n+1\rangle\langle n|\,, \tag{3.9}
$$

where f_n and d_n are arbitrary real coefficients and $\{|n\rangle\}$ is the basis provided by the number operator $\hat{N}$. The ansatz (3.8) leads, in the $\theta \to 0$ limit, to $\phi \sim \rho(r) z^{M-1}$ which corresponds, in ordinary space, to the usual cylindrically symmetric ansatz with a Higgs field phase $(M-1)\varphi$ [28].

Inserting ansatz (3.8)–(3.9) into eq. (3.7) one obtains the following recurrence relations

$$
\begin{aligned}
2\sqrt{p}\,d_{p-1} - d_{p-1}^2 - 2\sqrt{p+1}\,d_p + d_p^2 &= -\frac{|\kappa|}{\kappa}f_p^2\,, \\
d_p &= \sqrt{p+1} - \sqrt{p+M}\frac{f_p}{f_{p+1}}
\end{aligned}
\tag{3.10}
$$

which can be combined into the following recurrence relation for the $f_n's$ coefficients

$$
\begin{aligned}
f_1^2 &= \frac{M f_0^2}{1 - \frac{|\kappa|}{\kappa}f_0^2}\,, \\
f_{p+1}^2 &= \frac{(p+M)f_p^2}{1 - \frac{|\kappa|}{\kappa}f_p^2 + (p+M-1)f_{p-1}^2/f_p^2}\,, \qquad p \geq 1\,.
\end{aligned}
\tag{3.11}
$$

If, instead, we choose the anti selfdual case ($\alpha = 1$), the equations to solve read

$$
\begin{aligned}
D_z\hat{\phi} &= 0\,, \\
B &= -\frac{1}{2\kappa}\hat{\phi}\hat{\bar{\phi}}\,.
\end{aligned}
\tag{3.12}
$$

In this case, the appropriate ansatz is

$$
\begin{aligned}
\hat{\phi} &= \sqrt{\frac{2|\kappa|}{\theta}}\sum_{n=0}^{\infty} f_n|n+M-1\rangle\langle n|\,, \\
\hat{A}_z &= \frac{i}{\sqrt{\theta}}\sum_{n=0}^{\infty} d_n|n+1\rangle\langle n|
\end{aligned}
\tag{3.13}
$$

and the recurrence relations become

$$
\begin{aligned}
2\sqrt{p}\,d_{p-1} - d_{p-1}^2 - 2\sqrt{p+1}\,d_p + d_p^2 &= -\frac{|\kappa|}{\kappa} f_{p-1}^2\,, \\
d_{p+1} &= \sqrt{p+M} - \sqrt{p+1}\frac{f_{p+1}}{f_p}
\end{aligned}
\tag{3.14}
$$

combining to

$$
\begin{aligned}
f_1^2 &= \left(M - \frac{|\kappa|}{\kappa} f_0^2\right) f_0^2\,, \\
f_{p+1}^2 &= \frac{1}{p+1} f_p^2 \left(-\frac{|\kappa|}{\kappa} f_p^2 + 1 + p\frac{f_p^2}{f_{p-1}^2}\right), \qquad p \geq 1\,.
\end{aligned}
\tag{3.15}
$$

The flux of the solutions is given by

$$
\frac{\Phi}{2\pi} = \theta \,\mathrm{Tr}\, B = -\frac{|\kappa|}{\kappa} \sum_p f_p^2\,. \tag{3.16}
$$

Finite flux configurations correspond to solutions such that

$$
\lim_{p\to\infty} f_p = 0\,. \tag{3.17}
$$

The analysis of the asymptotic behavior of the recurrence relation (3.10) shows that, for large p,

$$
f_p^2 \longrightarrow \frac{1}{p^\beta} \tag{3.18}
$$

with β a real positive parameter to be determined. The flux can also be obtained directly from the expression of B as

$$
\frac{\Phi}{2\pi} = \lim_{p\to\infty} \left(d_p^2 - 2d_p\sqrt{p+1}\right). \tag{3.19}
$$

Using eqs. (3.10) and (3.14) one gets,

$$
\begin{aligned}
\frac{\Phi}{2\pi} &= M - 1 + \beta\,, \qquad \kappa < 0\,, \\
\frac{\Phi}{2\pi} &= -M + 1 - \beta\,, \qquad \kappa > 0\,.
\end{aligned}
\tag{3.20}
$$

The numerical study of the recurrence relations reveals that eq. (3.10) has solutions only for $\kappa < 0$ while eq. (3.14) has solutions only for $\kappa > 0$. Thus, as in the commutative case, solutions exist only for $\lambda < 0$, that is, only for an "attractive" interaction. Given an initial arbitrary value f_0^2 for the recurrence relation, all coefficients can be determined in such a way that they satisfy (3.18). Since θ does not enter explicitly in the recurrence relations, one can construct a whole family of solutions parametrized by θ (which appears as a factor, see eq. (3.9) or (3.13)).

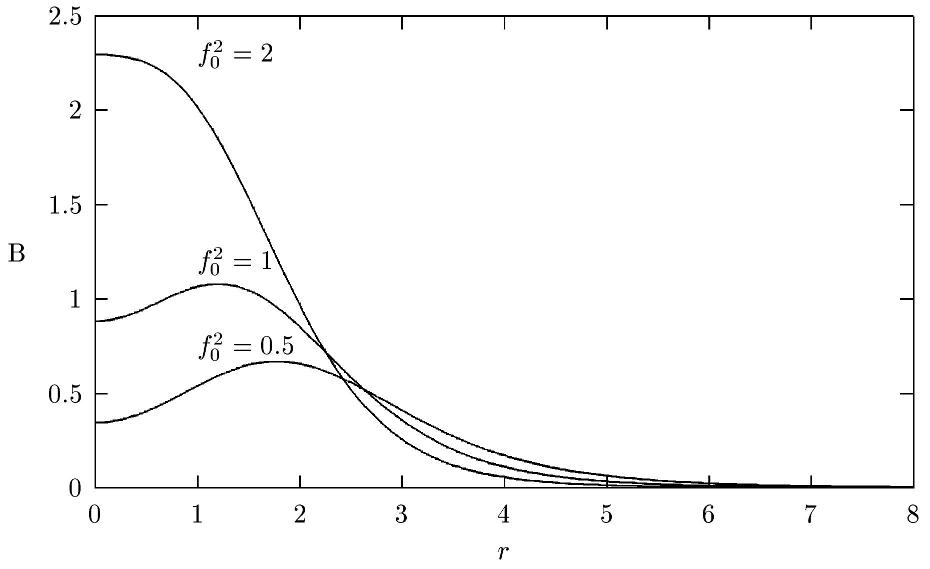

Figure 1: The magnetic field $B(r)$ for the selfdual non-relativistic solution for different values of f_0. θ and κ have been taken equal to 1.

We have explored numerically the whole $f_0^2 \geq 0$ range finding, for the selfdual case, that there exists, for every value of f_0, a consistent solution. In contrast, in the anti-selfdual case, the solution ceases to exist for $f_0^2 > M$. This is reminiscent of what happens for non-commutative Nielsen-Olesen anti-selfdual vortices [17].

We have find that β is a monotonically increasing function of f_0^2 such that

$$\lim_{f_0 \to 0} \beta = M + 1\,. \tag{3.21}$$

Now, according to eqs. (3.20) the magnetic flux of our exact solution is in general not quantized in non-commutative space.

We show in figures 1 and 2 the magnetic field B as a function of r, for the selfdual and anti-selfdual solutions, respectively, computed from eqs. (3.7) and (3.12). We plot different values of f_0^2 for a given θ. One sees that the B is reminiscent of the magnetic field corresponding to the ordinary (commutative) case, except that in the latter case $B(0) = 0$ while in the present non-commutative case $B(0) = B_0$, with B_0 positive (negative) for the selfdual (anti-selfdual) case.

Let us relate at this point this result with that corresponding to ordinary space. As originally shown in [28] Bogomol'nyi equations for the non-relativistic system can be exactly solved since the problem can be reduced to finding solutions to the Liouville equation. Then, the most general axially symmetric regular solution gives, for the selfdual case,

$$\begin{aligned} \phi^{\rm comm} &= \frac{2\sqrt{2|\kappa|}}{r} M \left(\left(\frac{r_0}{r}\right)^M + \left(\frac{r}{r_0}\right)^M \right)^{-1} \exp\left(i(M-1)\,\varphi\right)\,, \\ M &= 1, 2, \ldots\,, \end{aligned} \tag{3.22}$$

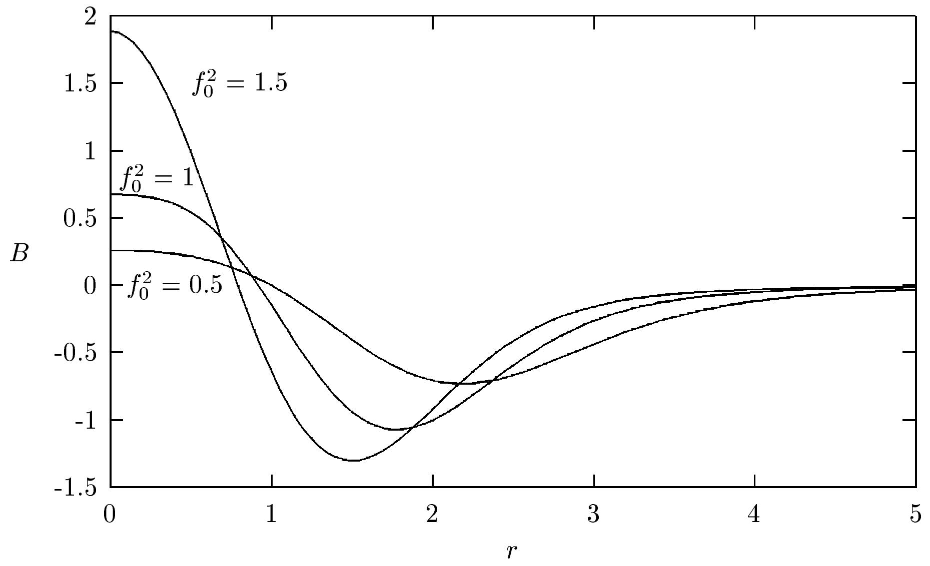

Figure 2: The magnetic field $B(r)$ for the anti-selfdual non-relativistic solution for different values of f_0. θ and κ have been taken equal to 1.

where r_0 is an integration constant. The other free parameter is M which is quantized on regularity grounds. Accordingly, the magnetic flux associated to this solution is quantized,

$$\Phi^{\rm comm} = 2\pi(2M)\,. \tag{3.23}$$

Note that the flux of our selfdual non-commutative solutions coincides, in the $f_0 \to 0$ limit with that of the ordinary case. To study the connection between our solution and that in ordinary space in more detail, let us consider the $\theta \to 0$ limit of the former in configuration space. It is enough to consider the small r region where the commutative solution (3.22) can be written in the form

$$\phi^{\rm comm} = \frac{2\sqrt{2|\kappa|}}{r_0^M} M z^{M-1} + O\left(r^{3M-1}\right). \tag{3.24}$$

Since, for small θ, $r^2 \approx \theta \hat{N}$, the leading contribution in the non-commutative case corresponds to the $n = 0$ term in solution (3.8),

$$\begin{aligned}\hat{\phi} &\approx \sqrt{\frac{2|\kappa|}{\theta}} f_0 |0\rangle\langle M-1| = \sqrt{\frac{2|\kappa|}{\theta}} f_0 |0\rangle\langle 0| \frac{a^{M-1}}{\sqrt{(M-1)!}} \\ &\approx \sqrt{\frac{2|\kappa|}{\theta}} f_0 |0\rangle\langle 0| \frac{z^{M-1}}{\sqrt{(2\theta)^{M-1}(M-1)!}}\,. \end{aligned} \tag{3.25}$$

A relation between r_0 and f_0 can be found comparing eqs. (3.24) and (3.25)

$$f_0^2 = 2^{M+1} M! M \left(\frac{\theta}{r_0^2}\right)^M. \tag{3.26}$$

If f_0^2 does not vanish as θ^M in the $\theta \to 0$ limit, the non-commutative solution goes in this limit to a singular solution in ordinary space. Only when the behavior (3.26) is satisfied, the $\theta \to 0$ limit converges to the Jackiw-Pi solution [28]. We have numerically checked this finding that, already for $\theta/r_0^2 \approx 0.01$, the non-commutative and the Jackiw-Pi solutions are indistinguishable.

4. BPS equations for the relativistic model

The associated hamiltonian for the model (2.17) for static field configurations is

$$H = \int d^2x \Big(\overline{D_i\phi} * D_i\phi + A_0 * A_0 * \phi * \bar{\phi} + V[\phi * \bar{\phi}] \Big) . \tag{4.1}$$

The Gauss law deriving from (2.17) takes, for static configurations, the form

$$2\kappa B = -\Big(\phi * \bar{\phi} * A_0 + A_0 * \phi * \bar{\phi} \Big) . \tag{4.2}$$

Assuming that A_0 (Moyal) commutes with $\phi * \bar{\phi}$ (as it will be the case for our ansatz, see below) we have

$$A_0 = -\kappa (\phi * \bar{\phi})^{-1} * B . \tag{4.3}$$

Inserting (4.3) in (4.1) and integrating by parts one gets

$$\begin{aligned} H = \int d^2x \Bigg(& \overline{(D_1 + i\alpha D_2)\phi} * (D_1 - i\alpha D_2)\phi + \\ & + \frac{\kappa^2}{|\phi|^2} * \left(B + \alpha \frac{1}{2\kappa^2} |\phi|^2 * (|\phi|^2 - v^2) \right)^2 \Bigg) - \alpha v^2 \Phi , \end{aligned} \tag{4.4}$$

where $\alpha = \pm 1$, $|\phi|^2 = \phi * \bar{\phi}$ and $\Phi = \int d^2x B$ is, as before, the magnetic flux. Thus, in the *selfdual case* ($\alpha = -1$) the energy is bounded by $v^2\Phi$, and the bound is saturated when the selfdual equations are fulfilled

$$\begin{aligned} D_{\bar{z}}\phi &= 0 , \\ B + \frac{1}{2\kappa^2} |\phi|^2 * (|\phi|^2 - v^2) &= 0 . \end{aligned} \tag{4.5}$$

Analogously, in the *anti-selfdual* case ($\alpha = 1$) the energy bound is $-v^2\Phi$ and is reached when the anti-selfdual equations are satisfied

$$\begin{aligned} D_z\phi &= 0 , \\ B - \frac{1}{2\kappa^2} |\phi|^2 * (|\phi|^2 - v^2) &= 0 . \end{aligned} \tag{4.6}$$

Let us analyze the selfdual case first. As in the non-relativistic case we shall work in the operator framework and propose an ansatz of the form

$$\begin{aligned} \hat{\phi} &= v \sum_{n=0}^{\infty} f_n |n\rangle\langle n + M| , \\ \hat{A}_z &= \frac{i}{\sqrt{\theta}} \sum_{n=0}^{\infty} d_n |n+1\rangle\langle n| , \end{aligned} \tag{4.7}$$

where again f_n and d_n are arbitrary real coefficients. With this ansatz both the magnetic field and $|\phi|^2$ are diagonal

$$B = \frac{1}{\theta}\left(\sum_{n=1}(g_{n-1} - g_m)\,|n\rangle\langle n| - g_0|0\rangle\langle 0|\right), \tag{4.8}$$

$$|\hat{\phi}|^2 = v^2\sum_{n=0} f_n^2|n\rangle\langle n|\,, \tag{4.9}$$

where $g_n = 2d_n\sqrt{n+1} - d_n^2$. Consequently, from the Gauss law (4.2), we see that A_0 commutes with $|\hat{\phi}|^2$ and can be solved as in eq. (4.3).

The selfdual system (4.5) is then equivalent to the following system of recurrence relations

$$\begin{aligned} d_p &= \sqrt{p+1} - \sqrt{p+M}\frac{f_p}{f_{p+1}}\,, \qquad p \geq 0\,, \\ f_1^2 &= \frac{(M+1)f_0^2}{1 + a\,f_0^2(1-f_0^2)}\,, \\ f_{p+1}^2 &= \frac{(p+M+1)f_p^2}{1 + a\,f_p^2(1-f_0^2) + (p+M)f_{p-1}^2/f_p^2}\,, \qquad p \geq 1\,, \end{aligned} \tag{4.10}$$

where $a = v^4\theta/(2\kappa^2)$.

For the anti-selfdual case, the ansatz we propose is

$$\begin{aligned} \hat{\phi} &= v\sum_{n=0}^{\infty} f_n|n+M\rangle\langle n|\,, \\ \hat{A}_z &= \frac{i}{\sqrt{\theta}}\sum_{n=0}^{\infty} d_n|n+1\rangle\langle n|\,. \end{aligned} \tag{4.11}$$

The magnetic field has the same form as in (4.9) while one has for the scalar field

$$|\hat{\phi}|^2 = v^2\sum_{n=0} f_n^2|n+M\rangle\langle n+M|\,. \tag{4.12}$$

The anti-selfdual recurrence relation take the form

$$\begin{aligned} d_{p+1} &= \sqrt{p+M+1} - \sqrt{p+1}\frac{f_{p+1}}{f_p}\,, \\ f_1^2 &= \left(M+1 - af_0^2(1-f_0^2)\right) f_0^2\,, \\ f_{p+1}^2 &= \frac{1}{p+1}f_p^2\left(1 - af_p^2(1-f_p^2) + p\frac{f_p^2}{f_{p-1}^2}\right), \qquad p \geq 1\,. \end{aligned} \tag{4.13}$$

We have studied systems (4.10) and (4.13) numerically. Given a value for f_0 one can then determine all f_n's from (4.10) or (4.13). The correct value for f_0 should make $f_n^2 \to 1$ asymptotically so that boundary conditions are satisfied (we are looking

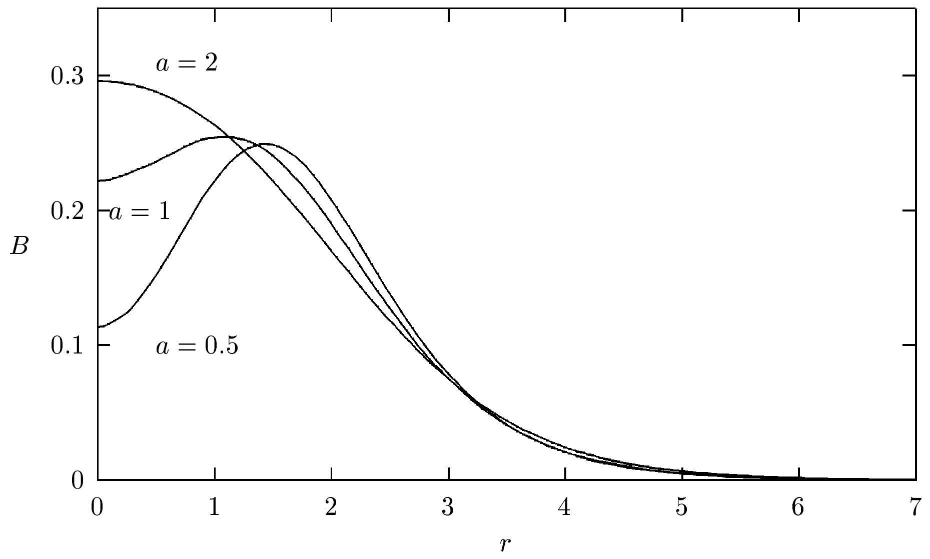

Figure 3: The magnetic field $B(r)$ for the selfdual relativistic solution for different values of $a = v^2\theta/(2\kappa^2)$.

for symmetry breaking solutions). The values of these coefficients will depend on the choice of the dimensionless parameter $a = \theta v^4/(2\kappa^2)$. We have explored the whole range of a and found a consistent solution for any positive integer M both in the selfdual and in the anti-selfdual case. In contrast with the non-relativistic model, there is only one value of f_0 leading to the appropriate boundary condition, for each value of a. For example, for the self-dual case we have

$$\begin{aligned} a &= 0.5\,, \qquad f_0^2 = 0.2168142\ldots\,, \\ a &= 1.0\,, \qquad f_0^2 = 0.4037747\ldots\,, \\ a &= 2.0\,, \qquad f_0^2 = 0.6228436\ldots\,. \end{aligned} \tag{4.14}$$

Once the $f_n's$ and $d_n's$ are determined in this way, the magnetic field can be computed using eq. (4.8) or Bogomol'nyi equation. As an example, for the selfdual case, one has

$$\hat{B} = \frac{v^2}{2\kappa^2}\sum_{n=0}^{\infty} f_n^2\left(1 - f_n^2\right)|n\rangle\langle n| \tag{4.15}$$

or, using the explicit formula for $|n\rangle\langle n|$ in configuration space [6]

$$B(r) = \frac{v^2}{\kappa^2}\sum_{n=0}^{\infty}(-1)^n f_n^2\left(1 - f_n^2\right)\exp\left(-\frac{r^2}{\theta}\right)L_n\left(2\frac{r^2}{\theta}\right), \tag{4.16}$$

where L_n are the Laguerre polynomials.

We show in figure 3 (selfdual case) and 4 (anti-selfdual case) the resulting magnetic field B as a function of r for different values of θ. For $\theta = 0$ we recover in both cases the CS vortex solutions found in [29]–[30]. One should note that, as in the non-

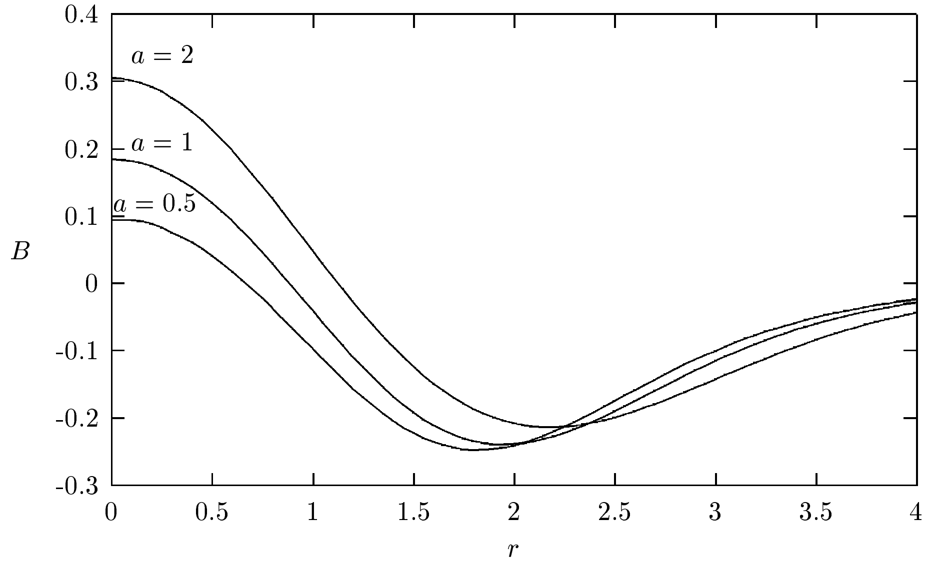

Figure 4: The magnetic field $B(r)$ for the anti-selfdual relativistic solution for different values of $a = v^2\theta/(2\kappa^2)$.

relativistic case, the magnetic field profile corresponding to the anti-selfdual case is not the trivial reverse of the selfdual one. This is related, as before, to the presence of the parity breaking parameter θ. As θ grows, the magnetic field differs more and more from the annulus-shaped ordinary CS vortex with a value at the origin which grows till $B(0)$ becomes a maximum. It is important to stress that we have found vortex solutions in the whole range of θ *both in the selfdual and anti-selfdual cases* in contrast with what happens for Nielsen-Olesen vortices where anti-selfdual solutions do not exist for θ larger than a critical value [17].

The magnetic flux of the solutions can be computed using

$$\Phi = 2\pi\theta \, \mathrm{Tr} \, \hat{B} \, . \tag{4.17}$$

One finds,

$$\Phi = 2\pi M \, , \qquad M = 1, 2, \ldots \tag{4.18}$$

showing that in the relativistic case, the magnetic flux is quantized for all θ. This, and the expression (4.4) for the hamiltonian allows to write the energy of the selfdual ($\alpha = -1$) and anti-selfdual ($\alpha = 1$) solitons in the form

$$H = (2\pi v^2) M \tag{4.19}$$

which coincides with the expression for CS solitons in ordinary space first found in [29]–[30]

JHEP02(2001)036

5. Conclusions

In summary, we have constructed exact soliton solutions to non-commutative Chern-Simons theory coupled to a charged scalar in 3-dimensional space-time. We have shown the existence of first order Bogomol'nyi equations and we have found, in the non-relativistic case, that an attractive ϕ^4 interaction guarantees, as in ordinary space, the existence of regular vortex-like solutions. There is however an important difference that manifests at finite θ: while non-topological ordinary solitons have, on regularity grounds, an associated quantized magnetic flux, non-commutative solitons can have arbitrary flux. Only in the $\theta \to 0$ limit, in which these solutions approach smoothly the ordinary ones, the flux becomes quantized. Remarkably, one can find a relation between the arbitrary integration constant arising in the solution of the Liouville equation satisfied by the Higgs field in ordinary space and the non-commutative parameter θ. It should be stressed that because of the presence of θ, anti-selfdual solutions can not be trivially obtained from selfdual ones by making $B \to -B$. In fact, we have shown that although solutions exist in both cases, there is a range of parameters where anti-selfdual solitons cease to exist. Concerning the relativistic case, as in ordinary space, a ϕ^6 potential guarantees the existence Bogomol'nyi equations and vortex-like topological solutions which also approach smoothly ordinary ones when $\theta \to 0$. Again, the presence of θ make selfdual and anti-selfdual solutions not trivially connected. Expressions for the magnetic flux and the energy of the CS solitons coincide with those in ordinary space for arbitrary θ.

As stressed in the introduction, one of the interests in CS solitons concerns their possible use in understanding relevant phenomena in planar physics. The connection between non-commutative field theories and systems in strong magnetic fields [33]–[34] make them attractive for a field theoretical approach to the Quantum Hall and Bohm-Aharonov effects [35]–[39]. We hope to report on these issues in a separate publication.

Acknowledgments

This work is partially supported by CICBA, CONICET (PIP 4330/96), ANPCYT (PICT 97/2285). G.S.L. and E.F.M. are partially supported by Fundación Antorchas.

References

[1] A. Connes, M.R. Douglas and A. Schwarz, *Noncommutative geometry and matrix theory: compactification on tori*, *J. High Energy Phys.* **02** (1998) 003 [hep-th/9711162].

[2] M.R. Douglas and C. Hull, *D-branes and the noncommutative torus*, *J. High Energy Phys.* **02** (1998) 008 [hep-th/9711165].

[3] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *J. High Energy Phys.* **09** (1999) 032 [hep-th/9908142].

[4] A. Hashimoto and K. Hashimoto, *Monopoles and dyons in non-commutative geometry*, *J. High Energy Phys.* **11** (1999) 005 [hep-th/9909202].

[5] S. Moriyama, *Noncommutative monopole from nonlinear monopole*, *Phys. Lett.* **B 485** (2000) 278 [hep-th/0003231].

[6] R. Gopakumar, S. Minwalla and A. Strominger, *Noncommutative solitons*, *J. High Energy Phys.* **05** (2000) 020 [hep-th/0003160].

[7] D.J. Gross and N. Nekrasov, *Monopoles and strings in noncommutative gauge theory*, *J. High Energy Phys.* **07** (2000) 034 [hep-th/0005204]; *Dynamics of strings in noncommutative gauge theory*, *J. High Energy Phys.* **10** (2000) 021 [hep-th/0007204]; *Solitons in noncommutative gauge theory*, hep-th/0010090.

[8] D.P. Jaktar, G. Mandal and S.R. Wadia, *Nielsen-Olesen vortices in noncommutative abelian Higgs model*, *J. High Energy Phys.* **09** (2000) 018 [hep-th/0007078].

[9] A.S. Gorsky, Y.M. Makeenko and K.G. Selivanov, *On noncommutative vacua and noncommutative solitons*, *Phys. Lett.* **B 492** (2000) 344 [hep-th/0007247].

[10] M. Aganagic, R. Gopakumar, S. Minwalla and A. Strominger, *Unstable solitons in noncommutative gauge theory*, hep-th/0009142.

[11] A.P. Polychronakos, *Flux tube solutions in noncommutative gauge theories*, *Phys. Lett.* **B 495** (2000) 407 [hep-th/0007043].

[12] N.A. Nekrasov, *Noncommutative instantons revisited*, hep-th/0010017.

[13] J.A. Harvey, P. Kraus and F. Larsen, *Exact noncommutative solitons*, *J. High Energy Phys.* **12** (2000) 024 [hep-th/0010060].

[14] M. Hamanaka and S. Terashima, *On exact noncommutative BPS solitons*, hep-th/0010221.

[15] K. Hashimoto, *Fluxons and exact BPS solitons in non-commutative gauge theory*, *J. High Energy Phys.* **12** (2000) 023 [hep-th/0010251].

[16] D. Bak, *Exact multi-vortex solutions in noncommutative abelian-Higgs theory*, *Phys. Lett.* **B 495** (2000) 251 [hep-th/0008204].

[17] D. Bak, K. Lee and J.-H. Park, *Noncommutative vortex solitons*, hep-th/0011099.

[18] G.S. Lozano, E.F. Moreno and F.A. Schaposnik, *Nielsen-Olesen vortices in noncommutative space*, hep-th/0011205.

[19] See E. Fradkin, *Field theories of condensed matter systems*, and references therein.

[20] See R.Jackiw, *Self-dual Chern-Simons solitons* in *Diverse topics in theoretical and mathematical physics* World Sci. Singapore, 1995, pp. 465–514 and references therein.

[21] C.-S. Chu, *Induced Chern-Simons and WZW action in noncommutative spacetime*, *Nucl. Phys.* **B 580** (2000) 352 [hep-th/0003007].

[22] A.A. Bichl, J.M. Grimstrup, V. Putz and M. Schweda, *Perturbative Chern-Simons theory on noncommutative R^3*, *J. High Energy Phys.* **07** (2000) 046 [hep-th/0004071].

[23] G.-H. Chen and Y.-S. Wu, *One-loop shift in noncommutative Chern-Simons coupling*, *Nucl. Phys.* **B 593** (2001) 562 [hep-th/0006114].

[24] S. Mukhi and N.V. Suryanarayana, *Chern-Simons terms on noncommutative branes*, *J. High Energy Phys.* **11** (2000) 006 [hep-th/0009101].

[25] N. Grandi and G.A. Silva, *Chern-simons action in noncommutative space*, hep-th/0010113.

[26] A.P. Polychronakos, *Noncommutative Chern-Simons terms and the noncommutative vacuum*, *J. High Energy Phys.* **11** (2000) 008 [hep-th/0010264].

[27] V. John, A.V. Nguyen and C.W. Kameshwar, *Chern-Simons terms in noncommutative geometry and its application to bilayer quantum hall systems*, *Phys. Lett.* **B 371** (1996) 252 [hep-th/9503091].

[28] R. Jackiw and S.-Y. Pi, *Soliton solutions to the gauged nonlinear schrodinger equation on the plane*, *Phys. Rev. Lett.* **64** (1990) 2969; *Reply to: "Comment on 'Soliton solutions to the gauged nonlinear Schrodinger equation on the plane'."* *Phys. Rev. Lett.* **66** (1991) 2682; *Classical and quantal nonrelativistic Chern-Simons theory*, *Phys. Rev.* **D 42** (1990) 3500.

[29] J. Hong, Y. Kim and P.Y. Pac, *On the multivortex solutions of the abelian Chern-Simons-Higgs theory*, *Phys. Rev. Lett.* **64** (1990) 2230.

[30] R. Jackiw and E.J. Weinberg, *Selfdual Chern-Simons vortices*, *Phys. Rev. Lett.* **64** (1990) 2234;
R. Jackiw, K. Lee and E.J. Weinberg, *Selfdual Chern-Simons solitons*, *Phys. Rev.* **D 42** (1990) 3488.

[31] H.J. de Vega and F.A. Schaposnik, *A classical vortex solution of the abelian Higgs model*, *Phys. Rev.* **D 14** (1976) 1100.

[32] E.B. Bogomol'nyi, *Stability of classical solutions*, *Sov. J. Nucl. Phys.* **24** (1976) 449.

[33] M.M. Sheikh-Jabbari, *Open strings in a B-field background as electric dipoles*, *Phys. Lett.* **B 455** (1999) 129 [hep-th/9901080].

[34] D. Bigatti and L. Susskind, *Magnetic fields, branes and noncommutative geometry*, *Phys. Rev.* **D 62** (2000) 066004 [hep-th/9908056].

[35] V. Pasquier, *Skyrmions in the quantum hall effect and noncommutative solitons*, *Phys. Lett.* **B 490** (2000) 258 [hep-th/0007176].

[36] J.H. Brodie, L. Susskind and N. Toumbas, *How Bob Laughlin tamed the giant graviton from Taub-Nut space*, *J. High Energy Phys.* **02** (2001) 003 [hep-th/0010105].

[37] S.S. Gubser and M. Rangamani, *D-brane dynamics and the quantum hall effect*, hep-th/0012155.

[38] G. Lozano, *Ground state energy for nonrelativistic bosons coupled to Chern-Simons gauge fields*, *Phys. Lett.* **B 283** (1992) 70;
O .Bergmann and G. Lozano, *Aharonov-Bohm scattering, contact interactions and scale invariance*, *Ann. Phys. (NY)* **229** (1994) 416 [hep-th/9302116].

[39] M. Chaichian, A. Demichev, P. Presnajder, M.M. Sheikh-Jabbari and A. Tureanu, *Aharonov-Bohm effect in noncommutative spaces*, hep-th/0012175.